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Conceptions of School Algebra and Uses of Variables

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WHAT IS SCHOOL ALGEBRA?

ALGEBRA is not easily defined. The algebra taught in school has quite a different cast from the algebra taught to mathematics majors. Two mathematicians whose writings have greatly influenced algebra instruction at the college level, Saunders Mac Lane and Garrett Birkhoff (1967), begin their *Algebra* with an attempt to bridge school and university algebras:

> Algebra starts as the art of manipulating sums, products, and powers of numbers. The rules for these manipulations hold for all numbers, so the manipulations may be carried out with letters standing for the numbers. It then appears that the same rules hold for various different sorts of numbers . . . and that the rules even apply to things . . . which are not numbers at all. An algebraic system, as we will study it, is thus a set of elements of any sort on which functions such as addition and multiplication operate, provided only that these operations satisfy certain basic rules. (P. 1)

If the first sentence in the quote above is thought of as arithmetic, then the second sentence is school algebra. For the purposes of this article, then, school algebra has to do with the understanding of "letters" (today we usually call them *variables*) and their operations, and we consider students to be studying algebra when they first encounter variables.

However, since the concept of variable itself is multifaceted, reducing algebra to the study of variables does not answer the question "What is school algebra?" Consider these equations, all of which have the same form—the product of two numbers equals a third:

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \cdot \tan x$
4. $1 = n \cdot (1/n)$
5. $y = kx$

Each of these has a different feel. We usually call (1) a formula, (2) an equation (or open sentence) to solve, (3) an identity, (4) a property, and (5) an equation of a function of direct variation (not to be solved). These different names reflect different uses to which the idea of variable is put. In (1), A, L, and W stand for the quantities area, length, and width and have the feel of knowns. In (2), we tend to think of x as unknown. In (3), x is an argument of a function. Equation (4), unlike the others, generalizes an arithmetic pattern, and n identifies an instance of the pattern. In (5), x is again an argument of a function, y the value, and k a constant (or parameter, depending on how it is used). Only with (5) is there the feel of "variability," from which the term *variable* arose. Even so, no such feel is present if we think of that equation as representing the line with slope k containing the origin.

Conceptions of variable change over time. In a text of the 1950s (Hart 1951a), the word *variable* is not mentioned until the discussion of systems (p. 168), and then it is described as "a changing number." The introduction of what we today call variables comes much earlier (p. 11), through formulas, with these cryptic statements: "In each formula, the letters represent numbers. *Use of letters to represent numbers is a principal characteristic of algebra*" (Hart's italics). In the second book in that series (Hart 1951b), there is a more formal definition of variable (p. 91): "A variable is a literal number that may have two or more values during a particular discussion."

Modern texts in the late part of that decade had a different conception, represented by this quote from May and Van Engen (1959) as part of a careful analysis of this term:

> Roughly speaking, a variable is a symbol for which one substitutes names for some objects, usually a number in algebra. A variable is always associated with a set of objects whose names can be substituted for it. These objects are called values of the variable. (P. 70)

Today the tendency is to avoid the "name-object" distinction and to think of a variable simply as a symbol for which things (most accurately, things from a particular replacement set) can be substituted.

The "symbol for an element of a replacement set" conception of variable seems so natural today that it is seldom questioned. However, it is not the only view possible for variables. In the early part of this century, the formalist school of mathematics considered variables and all other mathematics symbols merely as marks on paper related to each other by assumed or derived properties that are also marks on paper (Kramer 1981).

Although we might consider such a view tenable to philosophers but impractical to users of mathematics, present-day computer algebras such as MACSYMA and muMath (see Pavelle, Rothstein, and Fitch [1981]) deal with letters without any need to refer to numerical values. That is, today's computers can operate as both experienced and inexperienced users of algebra do operate, blindly manipulating variables without any concern for, or knowledge of, what they represent.

Many students think all variables are letters that stand for numbers. Yet the values a variable takes are not always numbers, even in high school mathematics. In geometry, variables often represent points, as seen by the use of the variables A, B, and C when we write "if $AB = BC$, then $\triangle ABC$ is isosceles." In logic, the variables p and q often stand for propositions; in analysis, the variable f often stands for a function; in linear algebra the variable A may stand for a matrix, or the variable $\mathbf{v}$ for a vector, and in higher algebra the variable $*$ may represent an operation. The last of these demonstrates that variables need not be represented by letters.

Students also tend to believe that a variable is always a letter. This view is supported by many educators, for

$$3 + x = 7 \text{ and } 3 + \triangle = 7$$

are usually considered algebra, whereas

$$3 + \_\_\_\_ = 7 \text{ and } 3 + ? = 7$$

are not, even though the blank and the question mark are, in this context of desiring a solution to an equation, logically equivalent to the x and the $\triangle$.

In summary, variables have many possible definitions, referents, and symbols. Trying to fit the idea of variable into a single conception oversimplifies the idea and in turn distorts the purposes of algebra.

TWO FUNDAMENTAL ISSUES IN ALGEBRA INSTRUCTION

Perhaps the major issue surrounding the teaching of algebra in schools today regards the extent to which students should be required to be able to do various manipulative skills by hand. (Everyone seems to acknowledge the importance of students having *some* way of doing the skills.) A 1977 NCTM-MAA report detailing what students need to learn in high school mathematics emphasizes the importance of learning and practicing these skills. Yet more recent reports convey a different tone:

> The basic thrust in Algebra I and II has been to give students moderate technical facility. . . . In the future, students (and adults) may not have to do much algebraic manipulation. . . . Some blocks of traditional drill can surely be curtailed. (CBMS 1983, p. 4)

A second issue relating to the algebra curriculum is the question of the role of functions and the timing of their introduction. Currently, functions are treated in most first-year algebra books as a relatively insignificant topic and first become a major topic in advanced or second-year algebra. Yet in some elementary school curricula (e.g., CSMP 1975) function ideas have been introduced as early as first grade, and others have argued that functions should be used as the major vehicle through which variables and algebra are introduced.

It is clear that these two issues relate to the very purposes for teaching and learning algebra, to the goals of algebra instruction, to the conceptions we have of this body of subject matter. What is not as obvious is that they relate to the ways in which variables are used. In this paper I try to present a framework for considering these and other issues relating to the teaching of algebra. My thesis is that the purposes we have for teaching algebra, the conceptions we have of the subject, and the uses of variables are inextricably related. ***Purposes for algebra*** *are determined by, or are related to, different* ***conceptions of algebra,*** *which correlate with the different relative importance given to various* ***uses of variables.***

Conception 1: Algebra as generalized arithmetic

In this conception, it is natural to think of variables as pattern generalizers. For instance, $3 + 5.7 = 5.7 + 3$ is generalized as $a + b = b + a$. The pattern

$$\begin{aligned} 3 \cdot 5 &= 15 \\ 2 \cdot 5 &= 10 \\ 1 \cdot 5 &= 5 \\ 0 \cdot 5 &= 0 \end{aligned}$$

is extended to give multiplication by negatives (which, in this conception, is often considered algebra, not arithmetic):

$$\begin{aligned} -1 \cdot 5 &= -5 \\ -2 \cdot 5 &= -10 \end{aligned}$$

This idea is generalized to give properties such as

$$-x \cdot y = -xy.$$

At a more advanced level, the notion of variable as pattern generalizer is fundamental in mathematical modeling. We often find relations between numbers that we wish to describe mathematically, and variables are exceedingly useful tools in that description. For instance, the world record T (in seconds) for the mile run in the year Y since 1900 is rather closely described by the equation

$$T = -0.4Y + 1020.$$

This equation merely generalizes the arithmetic values found in many almanacs. In 1974, when the record was 3 minutes 51.1 seconds and had not changed in seven years, I used this equation to predict that in 1985 the record would be 3 minutes 46 seconds (for graphs, see Usiskin [1976] or Bushaw et al. [1980]). The actual record at the end of 1985 was 3 minutes 46.31 seconds.

The key instructions for the student in this conception of algebra are *translate* and *generalize*. These are important skills not only for algebra but also for arithmetic. In a compendium of applications of arithmetic (Usiskin and Bell 1984), Max Bell and I concluded that it is impossible to adequately study arithmetic without implicitly or explicitly dealing with variables. Which is easier, "The product of any number and zero is zero" or "For all n, $n \cdot 0 = 0$"? The superiority of algebraic over English language descriptions of number relationships is due to the similarity of the two syntaxes. The algebraic description looks like the numerical description; the English description does not. A reader in doubt of the value of variables should try to describe the rule for multiplying fractions first in English, then in algebra.

Historically, the invention of algebraic notation in 1564 by François Viète (1969) had immediate effects. Within fifty years, analytic geometry had been invented and brought to an advanced form. Within a hundred years, there was calculus. Such is the power of algebra as generalized arithmetic.

Conception 2: Algebra as a study of procedures for solving certain kinds of problems

Consider the following problem:

> When 3 is added to 5 times a certain number, the sum is 40. Find the number.

The problem is easily translated into the language of algebra:

$$5x + 3 = 40$$

Under the conception of algebra as a generalizer of patterns, we do not have unknowns. We generalize known relationships among numbers, and so we do not have even the feeling of unknowns. Under that conception, this problem is finished; we have found the general pattern. However, under the conception of algebra as a study of procedures, we have only begun.

We solve with a procedure. Perhaps add -3 to each side:

$$5x + 3 + -3 = 40 + -3$$

Then simplify (the number of steps required depends on the level of student and preference of the teacher):

$$5x = 37$$

Now solve this equation in some way, arriving at $x = 7.4$. The "certain number" in the problem is 7.4, and the result is easily checked.

In solving these kinds of problems, many students have difficulty moving from arithmetic to algebra. Whereas the arithmetic solution ("in your head") involves subtracting 3 and dividing by 5, the algebraic form $5x + 3$ involves multiplication by 5 and addition of 3, the inverse operations. That is, to set up the equation, you must think precisely the opposite of the way you would solve it using arithmetic.

In this conception of algebra, variables are either *unknowns* or *constants*. Whereas the key instructions in the use of a variable as a pattern generalizer are translate and generalize, the key instructions in this use are *simplify* and *solve*. In fact, "simplify" and "solve" are sometimes two different names for the same idea: For example, we ask students to solve $|x - 2| = 5$ to get the answer $x = 7$ or $x = -3$. But we could ask students, "Rewrite $|x - 2| = 5$ without using absolute value." We might then get the answer $(x - 2)^2 = 25$, which is another equivalent sentence.

Polya (1957) wrote, "If you cannot solve the proposed problem try to solve first some related problem" (p. 31). We follow that dictum literally in solving most sentences, finding equivalent sentences with the same solution. We also simplify expressions so that they can more easily be understood and used. To repeat: simplifying and solving are more similar than they are usually made out to be.

Conception 3: Algebra as the study of relationships among quantities

When we write $A = LW$, the area formula for a rectangle, we are describing a relationship among three quantities. There is not the feel of an unknown, because we are not solving for anything. The feel of formulas such as $A = LW$ is different from the feel of generalizations such as $1 = n \cdot (1/n)$, even though we can think of a formula as a special type of generalization.

Whereas the conception of algebra as the study of relationships may begin with formulas, the crucial distinction between this and the previous conceptions is that, here, variables *vary*. That there is a fundamental difference between the conceptions is evidenced by the usual response of students to the following question:

> What happens to the value of $1/x$ as x gets larger and larger?

The question seems simple, but it is enough to baffle most students. We have not asked for a value of x, so x is not an unknown. We have not asked the student to translate. There is a pattern to generalize, but it is not a pattern that looks like arithmetic. (It is not appropriate to ask what happens to the value of 1/2 as 2 gets larger and larger!) It is fundamentally an algebraic pattern. Perhaps because of its intrinsic algebraic nature, some mathematics educators believe that algebra should first be introduced through this use of variable. For instance, Fey and Good (1985) see the following as the key questions on which to base the study of algebra:

> For a given function $f(x)$, find—
> 1. $f(x)$ for $x = a$;
> 2. x so that $f(x) = a$;
> 3. x so that maximum or minimum values of $f(x)$ occur;
> 4. the rate of change in f near $x = a$;
> 5. the average value of f over the interval (a,b). (P. 48)

Under this conception, a variable is an *argument* (i.e., stands for a domain value of a function) or a *parameter* (i.e., stands for a number on which other numbers depend). Only in this conception do the notions of independent variable and dependent variable exist. Functions arise rather immediately, for we need to have a name for values that depend on the argument or parameter x. Function notation (as in $f(x) = 3x + 5$) is a new idea when students first see it: $f(x) = 3x + 5$ looks and feels different from $y = 3x + 5$. (In this regard, one reason $y = f(x)$ may confuse students is because the function f, rather than the argument x, has become the parameter. Indeed, the use of $f(x)$ to name a function, as Fey and Good do in the quote above, is seen by some educators as contributing to that confusion.)

That variables as arguments differ from variables as unknowns is further evidenced by the following question:

Find an equation for the line through (6,2) with slope 11.

The usual solution combines all the uses of variables discussed so far, perhaps explaining why some students have difficulty with it. Let us analyze the usual solution. We begin by noting that points on a line are related by an equation of the form

$$y = mx + b.$$

This is both a pattern among variables and a formula. In our minds it is a function with domain variable x and range variable y, but to students it is not clear which of m, x, or b is the argument. As a pattern it is easy to understand, but in the context of this problem, some things are unknown. All the letters look like unknowns (particularly the x and y, letters traditionally used for that purpose).

Now to the solution. Since we know m, we substitute for it:

$$y = 11x + b$$

Thus m is here a constant, not a parameter. Now we need to find b. Thus b has changed from parameter to unknown. But how to find b? We use one pair of the many pairs in the relationship between x and y. That is, we select a value for the argument x for which we know y. Having to substitute a pair

of values for x and y can be done because $y = mx + b$ describes a general pattern among numbers. With substitution,

$$2 = 11 \cdot 6 + b,$$

and so $b = -64$. But we haven't found x and y even though we have values for them, because they were not unknowns. We have only found the unknown b, and we substitute in the appropriate equation to get the answer

$$y = 11x - 64.$$

Another way to make the distinction between the different uses of the variables in this problem is to engage quantifiers. We think: For all x and y, there exist m and b with $y = mx + b$. We are given the value that exists for m, so we find the value that exists for b by using one of the "for all x and y" pairs, and so on. Or we use the equivalent set language: We know the line is $\{(x,y)\colon y = mx + b\}$ and we know m and try to find b. In the language of sets or quantifiers, x and y are known as *dummy variables* because any symbols could be used in their stead. It is rather hard to convince students and even some teachers that $\{x\colon 3x = 6\} = \{y\colon 3y = 6\}$, even though each set is $\{2\}$. Many people think that the function f with $f(x) = x + 1$ is not the same as the function g with the same domain as f and with $g(y) = y + 1$. Only when variables are used as arguments may they be considered as dummy variables; this special use tends to be not well understood by students.

Conception 4: Algebra as the study of structures

The study of algebra at the college level involves structures such as groups, rings, integral domains, fields, and vector spaces. It seems to bear little resemblance to the study of algebra at the high school level, although the fields of real numbers and complex numbers and the various rings of polynomials underlie the theory of algebra, and properties of integral domains and groups explain why certain equations can be solved and others not. Yet we recognize algebra as the study of structures by the properties we ascribe to operations on real numbers and polynomials. Consider the following problem:

Factor $3x^2 + 4ax - 132a^2$.

The conception of variable represented here is not the same as any previously discussed. There is no function or relation; the variable is not an argument. There is no equation to be solved, so the variable is not acting as an unknown. There is no arithmetic pattern to generalize.

The answer to the factoring question is $(3x + 22a)(x - 6a)$. The answer could be checked by substituting values for x and a in the given polynomial and in the factored answer, but this is almost never done. If factoring were checked that way, there would be a wisp of an argument that here we are generalizing arithmetic. But in fact, the student is usually asked to check by multiplying the binomials, exactly the same procedure that the student has employed to get the answer in the first place. It is silly to check by repeating the process used to get the answer in the first place, but in this kind of problem students tend to treat the variables as marks on paper, without numbers as a referent. In the conception of algebra as the study of structures, the variable is little more than an arbitrary symbol.

There is a subtle quandary here. We want students to have the referents (usually real numbers) for variables in mind as they use the variables. But we also want students to be able to operate on the variables without always having to go to the level of the referent. For instance, when we ask students to derive a trigonometric identity such as $2\sin^2 x - 1 = \sin^4 x - \cos^4 x$, we do not want the student to think of the sine or cosine of a specific number or even to think of the sine or cosine functions, and we are not interested in ratios in triangles. We merely want to manipulate $\sin x$ and $\cos x$ into a different form using properties that are just as abstract as the identity we wish to derive.

In these kinds of problems, faith is placed in properties of the variables, in relationships between x's and y's and n's, be they addends, factors, bases, or exponents. The variable has become an arbitrary object in a structure related by certain properties. It is the view of variable found in abstract algebra.

Much criticism has been leveled against the practice by which "symbol pushing" dominates early experiences with algebra. We call it "blind" manipulation when we criticize; "automatic" skills when we praise. Ultimately everyone desires that students have enough facility with algebraic symbols to deal with the appropriate skills abstractly. The key question is, What constitutes "enough facility"?

It is ironic that the two manifestations of this use of variable—theory and manipulation—are often viewed as opposite camps in the setting of policy toward the algebra curriculum, those who favor manipulation on one side, those who favor theory on the other. They come from the same view of variable.

VARIABLES IN COMPUTER SCIENCE

Algebra has a slightly different cast in computer science from what it has in mathematics. There is often a different syntax. Whereas in ordinary algebra, $x = x + 2$ suggests an equation with no solution, in BASIC the same sentence conveys the replacement of a particular storage location in a computer by a number two greater. This use of variable has been identified by Davis, Jockusch, and McKnight (1978, p. 33):

> Computers give us another view of the basic mathematical concept of variable. From a computer point of view, the name of a variable can be thought of as the address of some specific memory register, and the value of the variable can be thought of as the contents of this memory register.

In computer science, variables are often identified strings of letters and numbers. This conveys a different feel and is the natural result of a different setting for variable. Computer applications tend to involve large numbers of variables that may stand for many different kinds of objects. Also, computers are programmed to manipulate the variables, so we do not have to abbreviate them for the purpose of easing the task of blind manipulation.

In computer science the uses of variables cover all the uses we have described above for variables. There is still the generalizing of arithmetic. The study of algorithms is a study of procedures. In fact, there are typical algebra questions that lend themselves to algorithmic thinking:

> Begin with a number. Add 3 to it. Multiply it by 2. Subtract 11 from the result. . . .

In programming, one learns to consider the variable as an argument far sooner than is customary in algebra. In order to set up arrays, for example, some sort of function notation is needed. And finally, because computers have been programmed to perform manipulations with symbols without any referents for them, computer science has become a vehicle through which many students learn about variables (Papert 1980). Ultimately, because of this influence, it is likely that students will learn the many uses of variables far earlier than they do today.

SUMMARY

The different conceptions of algebra are related to different uses of variables. Here is an oversimplified summary of those relationships:

Conception of algebra	*Use of variables*
Generalized arithmetic	Pattern generalizers (translate, generalize)
Means to solve certain problems	Unknowns, constants (solve, simplify)
Study of relationships	Arguments, parameters (relate, graph)
Structure	Arbitrary marks on paper (manipulate, justify)

Earlier in this article, two issues concerning instruction in algebra were mentioned. Given the discussion above, it is now possible to interpret these issues as a question of the relative importance to be given at various levels of study to the various conceptions.

For example, consider the question of paper-and-pencil manipulative skills. In the past, one had to have such skills in order to solve problems and in order to study functions and other relations. Today, with computers able to simplify expressions, solve sentences, and graph functions, what to do with manipulative skills becomes a question of the importance of algebra as a structure, as the study of arbitrary marks on paper, as the study of arbitrary relationships among symbols. The prevailing view today seems to be that this should not be the major criterion (and certainly not the only criterion) by which algebra content is determined.

Consider the question of the role of function ideas in the study of algebra. It is again a question of the relative importance of the view of algebra as the study of relationships among quantities, in which the predominant manifestation of variable is as argument, compared to the other roles of algebra: as generalized arithmetic or as providing a means to solve problems.

Thus some of the important issues in the teaching and learning of algebra can be crystallized by casting them in the framework of conceptions of algebra and uses of variables, conceptions that have been changed by the explosion in the uses of mathematics and by the omnipresence of computers. No longer is it worthwhile to categorize algebra solely as generalized arithmetic, for it is much more than that. Algebra remains a vehicle for solving certain problems but it is more than that as well. It provides the means by which to describe and analyze relationships. And it is the key to the characterization and understanding of mathematical structures. Given these assets and the increased mathematization of society, it is no surprise that algebra is today the key area of study in secondary school mathematics and that this preeminence is likely to be with us for a long time.

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CAN YOUR ALGEBRA CLASS SOLVE THIS?

Problem 3. Find all real values of x that satisfy

$$(x^2 - 5x + 5)^{x^2-9x+20} = 1.$$

Solution on page 248

CAN YOUR ALGEBRA CLASS SOLVE THIS?

Problem 4. If p pencils cost c cents, how many pencils can be purchased for d dollars?

Solution on page 248